Universal amplitude combinations for self-avoiding walks, polygons and trails

John L Cardy†§ and Anthony J Guttmann‡∥

† Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 0EH, UK
‡ Department of Physics Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

Abstract. We give exact relations for a number of amplitude combinations that occur in the study of self-avoiding walks, polygons and lattice trails. In particular, we elucidate the lattice-dependent factors which occur in those combinations which are otherwise universal, show how these are modified for oriented lattices, and give new results for amplitude ratios involving even moments of the area of polygons. We also survey numerical results for a wide range of amplitudes on a number of oriented and regular lattices, and provide some new ones.

1. Introduction

In this paper we consider a number of amplitude combinations that are universal, up to explicit lattice-dependent factors, in the context of the $N \to 0$ limit of the $O(N)$ model, which describes self-avoiding walks and polygons. We correct and extend a table of such values given previously by one of us [1], and correct and generalize an exact expression for a particular amplitude combination given by the other [2]. We also point out an apparent error in another amplitude combination given previously [3]. We make clear the role that lattice-dependent factors (such as the number of sites per unit area) play in some of these otherwise universal combinations. In particular, we point out for the first time a distinction between oriented and unoriented lattices. In order to confirm these results, we have generated short series for the radius of gyration of self-avoiding polygons on the triangular and honeycomb lattices. We also give arguments for the universality and lattice independence of amplitude ratios involving even moments of the area of self-avoiding polygons.

The functions we are considering are: (i) the chain generating function for SAWs, $C(x) = \Sigma c_n x^n$; (ii) the corresponding polygon generating function, $P(x) = \Sigma p_n x^n$; (iii) the generating function for lattice trails, $T(x) = \Sigma t_n x^n$; (iv) the generating function for dumb-bell graphs $\Delta(x) = \Sigma d_n x^n$; (v) the mean-square end-to-end distance of n-step self-avoiding walks (SAWs) $\langle R_e^2 \rangle_n$; (vi) the mean-square radius of gyration of n-step polygons $\langle R^2 \rangle_n$; (vii) the mean-square radius of gyration of n-step SAWs $\langle R_g^2 \rangle_n$; (viii) moments of the area of polygons of perimeter n, $\langle a^p \rangle_n$; and (ix) the mean-square distance of a monomer from the origin of n-step SAWs $\langle R_m^2 \rangle_n$. In the above generating functions, c_n, p_n, t_n and d_n denote the total number of n-step SAWs, polygons, trails and dumb-bells. For SAWs and trails, an

§ Permanent address: Department of Physics, University of California, Santa Barbara CA 93106, USA.
∥ Permanent address: Department of Mathematics, University of Melbourne, Parkville, Vic 3052, Australia.

origin is chosen on an infinite lattice, and all distinct SAWs and trails are enumerated. For dumb-bells and polygons, we adopt the normal convention in which the total number *per lattice site* is given. As we discuss below, this is different from the number of distinct unrooted polygons (up to translations) on certain lattices, such as the honeycomb lattice. This distinction is one source of error in [1].

In terms of the quantities defined above, we denote the relevant amplitudes as follows:

$$c_n = A\mu^n n^{\gamma-1}[1+o(1)]$$

$$p_n = B\mu^n n^{\alpha-3}[1+o(1)]$$

$$t_n = H\mu^n n^{\gamma-1}[1+o(1)]$$

$$d_n = J\mu^n n^{\gamma-1}[1+o(1)]$$

(where the growth exponent μ for trails applies only to lattices of coordination number three [4]), and

$$\langle R_e^2\rangle_n = Cn^{2\nu}[1+o(1)]$$

$$\langle R^2\rangle_n = Dn^{2\nu}[1+o(1)]$$

$$\langle a^p\rangle_n = E^{(p)}n^{2p\nu}[1+o(1)] \tag{1}$$

$$\langle R_g^2\rangle_n = Fn^{2\nu}[1+o(1)]$$

$$\langle R_m^2\rangle_n = Gn^{2\nu}[1+o(1)]$$

where $\gamma = 43/32$, $\alpha = \frac{1}{2}$, $\nu = \frac{3}{4}$ and $\mu = x_c^{-1}$, the reciprocal of the critical point. Amplitudes B, D and $E^{(p)}$, which relate to polygons, will be zero for odd-order terms on loosely packed lattices, and will be non-zero only for every fourth term on certain lattices such as the L-lattice (an oriented square lattice in which every step must be followed by a step perpendicular to the preceding step) and the Manhattan lattice. We should remind the reader that the above asymptotic forms, and the exact values of the critical exponents, are assumptions in the sense that they have not been proven rigorously. Nevertheless they all follow from the central assumption that these problems have a continuum limit which corresponds to a particular exactly soluble field theory, and it is this correspondence on which shall base our analytic results. The appearance of the same growth exponent μ in each of the above has been rigorously demonstrated, however.

In the next section we generalize and correct the first amplitude relation, which gives the value of BD. We subsequently discuss the other known relations, and correct the relation for the combination BC. We then use these to provide amplitude estimates for some cases in which they have not been directly estimated.

2. The combination BD

Rather than giving the modifications to the argument in [2] (which, in fact, contains several errors of factors of 2), it is clearer to repeat the whole argument for a general lattice. The universality of the combination BD follows from the integral form [5] of Zamolodchikov's c-theorem [6], which reads

$$c = 3\pi t^2 \nu^{-2} \int r^2 \langle E(r)E(0)\rangle_c \, \mathrm{d}^2 r \tag{2}$$

where $E(r)$ is the energy density, which enters the Hamiltonian in the form $t \int E(r) d^2 r$, and v is the usual critical exponent governing the divergence of the correlation length. We wish to apply this to the $O(N)$ model whose lattice Hamiltonian is $\mathcal{H} = -x \sum_{\text{links}} E_{\text{lat}}(r)$ where r now labels links, and $E_{\text{lat}}(r) = s(r_<) \cdot s(r_>)$. Here $s(r_<)$ and $s(r_>)$ are $O(N)$ spins located at the sites $r_<$ and $r_>$ at the ends of the link at r, ordered in some standard fashion. In order to apply equation (2) correctly, we must relate the continuum energy density $E(r)$ to its lattice counterpart. This is done by equating the continuum and lattice Hamiltonians, so that

$$t \int E(r) \, d^2 r \longrightarrow (x_c - x) \sum_r E_{\text{lat}}(r). \tag{3}$$

It is convenient to work temporarily on a very large but finite lattice of total area $\mathcal{A}$, and to rewrite equation (2), using translational invariance, as

$$c = 3\pi t^2 v^{-2} \mathcal{A}^{-1} \int \int (r - r')^2 \langle E(r) E(r') \rangle_c \, d^2 r \, d^2 r'.$$

It is then apparent from equation (3) that we may simply replace this by its lattice version

$$c = 3\pi (x_c - x)^2 v^{-2} \mathcal{A}^{-1} \sum_{r, r'} (r - r')^2 \langle E_{\text{lat}}(r) E_{\text{lat}}(r') \rangle_c. \tag{4}$$

This should be valid in the scaling limit as $x \to x_c$, when the integral will be dominated by values of $|r - r'|$ on the scale of the correlation length ξ, which is much larger than the lattice spacing, so that the continuum approximation becomes arbitrarily accurate.

The next step is to evaluate the right-hand side of equation (4) in the limit $N \to 0$ as a sum over pairs of mutually SAWs connecting $(r_<, r_>)$ and $(r'_<, r'_>)$. These pairs of walks are then identified with self-avoiding polygons where the links at r and r' are marked. The contribution of a given polygon of length n, whose links are at $(r_1, r_2, \ldots, r_n)$, to the sum in equation (4) is then

$$N x^{n-2} {\sum_{k,l}}' (r_k - r_l)^2 \tag{5}$$

where the prime on the sum indicates that adjacent links are to be excluded. This excludes contributions where, for example, $r_>$ coincides with $r'_<$. Such contributions are certainly included in the right-hand side of equation (4), but they are more complicated to evaluate. Fortunately they are expected to be negligible in the scaling limit. It is worth checking the normalization in equation (5) for small polygons. For example, for $n = 4$ on a square lattice, the sum on the right-hand side of equation (4) gives $2N\mathcal{N}_b x^2$, where $\mathcal{N}_b$ is the total number of links, while equation (5) gives $4Nx^2$ for each elementary square, of which there are $\mathcal{N}_b/2$. It is also straightforward to check this for larger polygons. Now, if the sum in equation (5) were unrestricted, it would be equal to $2n^2 R_b^2$, where R_b^2 is the link-weighted squared radius of gyration of the polygon (which differs from the corresponding site-weighted quantity by terms of $O(1)$ as $n \to \infty$.) However, the effect of the restriction is only at the level of subleading terms down by one power of n, and is negligible in the scaling limit. If now we denote the mean square radius of gyration over all polygons of perimeter n by $\langle R^2 \rangle_n$, and the total number of such polygons by $\mathcal{N}_s p_n$, where $\mathcal{N}_s$ is the total number of sites on our large lattice, then, in the limits $N \to 0$ and $x \to x_c$,

$$c(N) \sim 6\pi N (x_c - x)^2 v^{-2} \left(\frac{\mathcal{N}_s}{\mathcal{A}} \right) \sum_n n^2 p_n \langle R^2 \rangle_n x^{n-2}. \tag{6}$$

This implies that the sum on the right-hand side has a singularity of the form $(x_c - x)^{-2}$. Since each term in these series is non-negative, this singularity on the positive real axis gives the radius of convergence, and, for close-packed lattices, we assume that there are no other singularities on $|x| = x_c$. Thus this singularity solely dictates the behaviour of the terms as $n \to \infty$. For loose-packed and oriented lattices, however, p_n is non-zero only if n is divisible by an integer σ. In this case the generating function is invariant under rotations $x \to x e^{2\pi i/\sigma}$, so that there must exist singularities of equal strengths at $x = x_c e^{2\pi i r/\sigma}$ where $r = 0, 1, \ldots, \sigma - 1$. Thus, the large n behaviour of p_n, for n divisible by σ, is greater by a factor σ than that expected on the basis of the singularity described by equation (6).

We conclude that

$$\lim_{n \to \infty} n p_n \langle R^2 \rangle_n x_c^n = \sigma a_0 \, c'(0) v^2 / 6\pi$$

where we have introduced the area per site $a_0 = \mathcal{A}/\mathcal{N}_s$. From the known values [7] $c = 1 - 6/m(m+1)$ and $v = (m+1)/4$, where $N = 2\cos(\pi/m)$, for the O(N) model, the second numerical factor is $5/32\pi^2$, so that, in terms of the amplitudes defined previously, we have

$$BD = \frac{5}{32\pi^2} \sigma a_0. \tag{7}$$

For the square lattice, $\sigma = 2$ and the area per site (measured in units of square lattice spacings) is unity. This then gives the result $5/16\pi^2$ found previously [2], which agrees with results from enumerations given in [8], as well as later data summarized in the final section.

For a general periodic lattice, we may write the factor $a_0 = \mathcal{A}/\mathcal{N}_s$ as v/κ, where v is the area of the unit cell, and κ counts the the number of sites per unit cell. In the above, we have defined p_n in the conventional way as the number of n-step polygons per site. If instead, as in [3], it is defined as the number of *distinct* n-step unrooted polygons (where two such polygons are regarded as being equivalent if they are related by translation by a lattice vector) then we may associate with each such polygon a unique site. This may be done, for example, by ordering the sites in a systematic manner, and associating with a given polygon the lowest site according to this ordering. In this way of counting, $\mathcal{N}_s$ should be replaced by the total number of sites which are equivalent to each other by lattice translations. These lattice translations divide the sites into equivalence classes, and there is just one representative of each equivalence class corresponding to each unit cell. Thus, with this definition of p_n, the factor κ does not enter the formula for BD. This is most easily seen by considering polygons on the honeycomb lattice. The first term in the generating function is $p_6 = \frac{1}{2}$ if the normalization is per site, and then $\kappa = 2$. Alternatively, $p_6 = 1$ gives the number of 6-step unrooted polygons.

This result may easily be generalized to polygons on oriented lattices. In this case each link r has a specified orientation, say from $r_<$ to $r_>$. Oriented walks on such a lattice are described by the $N \to 0$ limit of a complex O(N) spin model, where now the link energy is $E^{\text{lat}}(r) = s(r_>)^* \cdot s(r_<)$. Proceeding as before, the sum on the right-hand side of equation (4) is now equal to

$$N \sum_n x^{n-2} \sum_{\substack{\text{all } n\text{-step} \\ \text{oriented polygons}}} \sideset{}{'}\sum_{k,l} (r_k - r_l)^2$$

so that the appropriate lattice-dependent factor is the total number of n-step oriented polygons divided by the total area. If we define p_n as the number of n-step oriented

polygons per site, then this factor is equal to p_n/a_0, as before. However, the complex $O(N)$ model is equivalent to a real $O(2N)$ model, and therefore the appropriate value of the central charge on the left-hand side of equation (6) is $c(2N)$. This leads to an extra factor of 2 on differentiating with respect to N in the $N \to 0$ limit, so the final result for oriented lattices is $BD/\sigma a_0 = 5/16\pi^2$, as compared with equation (7) for the unoriented case. If we define p_n as the number of distinct unrooted oriented polygons, as in [3], then a_0 should be replaced by v, the unit cell size, in the above formula.

The results for unoriented and oriented lattices may be unified by supposing that in the unoriented case each link may be oriented in either direction, and by defining p_n in both cases in terms of the number of oriented polygons. The values of p_n for unoriented lattices will then be doubled, since each polygon on such a lattice has precisely two orientations. The result above will then apply to both types of lattice, and, presumably, to partially oriented lattices also. It makes physical sense, because if one looks at polygons with a fixed R^2, then the number of oriented polygons on an oriented lattice should be asymptotically the same as the number of such polygons on an unoriented, or partially oriented, lattice.

3. Polygon area amplitudes

Next we consider the universality of amplitudes involving the area of polygons. Our approach is as follows: let dl be an infinitesimal element of the curve formed by the polygon. Then, by Stokes' theorem, the area within the polygon is proportional to $\oint r \times dl$. We can write this equivalently as $\oint (r \times J) \, dl$, where J is a unit current flowing through the links of the polygon. Thus, moments of the area distribution are related to integrals over correlation functions of J. Such a current will be conserved in the ensemble of polygons. In the continuum field theory, it turns out that it is possible to identify the corresponding current. Conserved currents in field theory enjoy the special property that their correlation functions scale according to naive dimensional counting arguments. Moreover, the normalization of this particular current is fixed by considering its behaviour in the larger ensemble including SAWs with ends. By definition, each end will be a unit source or sink for this current. Thus the functional form, and overall normalization, of these correlation functions are, in principle, completely determined in the field theory.

We now put some flesh on these remarks. Consider a given oriented polygon, and let $J^{\text{lat}} = (J_1^{\text{lat}}, J_2^{\text{lat}})$ be a unit current flowing along the links, each of length b and labelled by r, in the direction of their orientation. Then the sum

$$a = \frac{1}{2}b \sum_r \epsilon_{ij} r_i J_j^{\text{lat}}$$

where ϵ_{ij} is the totally antisymmetric symbol, gives the *signed* area of the polygon (that is the area is positive for anticlockwise orientation, negative for clockwise orientation.) The mean signed area over all polygons with n steps is zero, since opposite orientations of a given polymer give an equal and opposite contribution. However, the mean square area, and, indeed, all the higher even moments, are non-zero and the same as for unoriented polygons. The squared area of a given polygon is

$$a^2 = \frac{1}{4}b^2 \sum_{r,r'} (r_i r_i' J_j^{\text{lat}}(r) J_j^{\text{lat}}(r') - r_i r_j' J_i^{\text{lat}}(r) J_j^{\text{lat}}(r')).$$

Using $\sum_r J_i^{\text{lat}}(r) = 0$ and $\sum_r r_i J_i^{\text{lat}}(r) = 0$, the right-hand side may be rewritten as $-\frac{1}{8}b^2 \sum_{r,r'}(r - r')^2 J_j^{\text{lat}}(r) J_j^{\text{lat}}(r')$.

Oriented self-avoiding polygons and walks are described by a complex $O(N)$ spin model in the limit $N \to 0$. The lattice degrees of freedom are complex $O(N)$ spins, and the Hamiltonian is $\mathcal{H} = -x \sum_r s(r_>)^* \cdot s(r_<) + \text{CC}$. Within this model, we may identify

$$J^{\text{lat}}(r) = xb^{-1}(r_> - r_<)(s(r_>)^* \cdot s(r_<) - s(r_<)^* \cdot s(r_>)).$$

Thus, if $\langle a^2 \rangle_n$ denotes the mean square area of oriented polygons of n steps, and, as before, $2N_s p_n$ is the number of such polygons (the factor of 2 is correct on an unoriented lattice, if p_n counts the number of unoriented polygons per site),

$$2N_s \sum_n p_n \langle a^2 \rangle_n x^n = -\frac{1}{8}b^2 \sum_{r,r'}(r - r')^2 \langle J^{\text{lat}}(r) \cdot J^{\text{lat}}(r') \rangle. \tag{8}$$

In the continuum limit, J^{lat} is replaced by a conserved current J, which is the Noether current for the $U(1)$ transformations in the complex $O(N)$ field theory which commute with the $O(N)$ rotations [9]. The connection between them is simply

$$\sum_r J^{\text{lat}}(r) \longrightarrow \int J(r)\, d^2 r$$

whereby the double sum over r and r' in equation (8) becomes a double integral, which, by translational invariance, equals

$$-\frac{1}{8}Ab^2 \int r^2 \langle J(r) \cdot J(0) \rangle\, d^2 r. \tag{9}$$

Now the main point is that the normalization of J in the continuum limit is fixed by the requirement that the unit source for this current is the free end of an oriented SAW [9]. This means that if $\phi(r)$ is the continuum field corresponding to $s(r)$ (suppressing $O(N)$ indices), representing the magnetization density of the $O(N)$ model, then

$$\int J(r') \cdot dS_{r'} \phi(r) = \phi(r)$$

$$\int J(r') \cdot dS_{r'} \phi(r)^* = -\phi(r)^* \tag{10}$$

where the integral is over the boundary of a small neighbourhood of r, and $dS_{r'}$ is the outward pointing normal. Equation (10) is supposed to make sense when inserted into correlation functions with other operators. It specifies [9] the normalization of the pole term in the short-distance expansion of J with ϕ and ϕ^*.

We see from equation (10) that J has unit scaling dimension (since J is a conserved current which generates a symmetry of the theory, it has no anomalous dimension.) The fact that the normalization of J is fixed, and therefore does not contain any metric factors, implies that its two-point function has the universal form

$$\langle J(r) \cdot J(0) \rangle = \xi^{-2} f(r/\xi) \tag{11}$$

where ξ is the correlation length. Therefore the integral in equation (9) is of the form $U\xi^2 \sim U\xi_0^2(1 - x/x_c)^{-2\nu}$, where U is a universal number, in principle calculable from the

underlying continuum field theory, and ξ_0 is the non-universal metric factor in the correlation length.

It then follows from equations (8) and (11) that

$$p_n \langle a^2 \rangle_n \sim \sigma a_0 U' \xi_0^2 \, n^{2\nu-1} x_c^{-n}$$

where U' is universal. Now we know that $\langle R^2 \rangle_n$ is related to the ratio of the second to the zeroth moments of the energy–energy correlation function in the $O(N)$ model, which is of the form $U'' \xi^2$, where again U'' is universal. Therefore the amplitude D defined in equation (1) has the form $U'' \xi_0^2$. Also $p_n \sim B n^{-2\nu-1} x_c^{-n}$. Combining these, we find

$$\frac{\langle a^2 \rangle_n}{\langle R^2 \rangle_n^2} \sim b^2 U''' \frac{\sigma a_0}{BD}.$$

But we argued above that the combination $BD/\sigma a_0$ should be universal and lattice independent. This establishes the universality and lattice independence of the ratio on the left-hand side, provided the area is measured in square lattice spacings.

If we now consider the higher even moments of the area, the argument generalizes straightforwardly. Schematically,

$$\mathcal{N}_s \sum_n p_n \langle a^p \rangle_n x^n \sim b^p \int \cdots \int r_1 \dots r_p \langle J(r_1) \dots J(r_p) \rangle \, d^2 r_1 \dots d^2 r_p$$

where all indices have been suppressed. The integral on the the the right-hand side is equal to $\mathcal{A} U^{(p)} \xi^{2p-2}$, where $U^{(p)}$ is universal. Following the argument through, we then discover that $\langle a^p \rangle_n / \langle R^2 \rangle_n^p$ should be universal, with no lattice-dependent factors.

However, it appears difficult to extend this result to odd moments of the unsigned area. This is because these quantities appear not to have any simple expression in terms of correlation functions in the continuum field theory. Nevertheless, in [10] it was argued that the corresponding ratio involving the first moment of the area $\langle a^1 \rangle_n / \langle R^2 \rangle_n = E^{(1)}/D$ should be universal, with no lattice-dependent factors. This is to be expected on physical grounds [11], since if lattice polygons are regarded as a model for two-dimensional vesicles, the area couples to the pressure p. Since p and $\langle R^2 \rangle$ are macroscopically measurable quantities, one might expect a universal relationship between them independent of the particular lattice model. Nevertheless, a theoretical demonstration of this is so far lacking. In [12], an argument was made for the finiteness of ratio $E^{(1)}/D$, but this does not prove its universality. Note that [11] the fact that moments of the area satisfy the inequalities

$$\langle a^1 \rangle \leqslant \langle a^2 \rangle^{1/2} \leqslant \langle a^3 \rangle^{1/3} \leqslant \cdots$$

together with our results for the even moments, does imply that the higher odd moments $\langle a^p \rangle$ do scale as expected for $p \geqslant 3$, but it does not imply that their ratios are universal or lattice independent. As pointed out [11], it is essential that area be measured in Euclidean units such that the lattice spacing is the same in calculating the radius of gyration, the mean area, and its moments.

4. Numerical results

Another universal amplitude ratio, first given in [13] and corrected in [14], is

$$\left(2 + \frac{y_t}{y_h}\right) \frac{F}{C} - \frac{2G}{C} + \frac{1}{2} = 0.$$

The individual quotients F/C and G/C are also universal [13]. The best estimates of these quantities is given in [14], as $F/C = 0.140\,29 \pm 0.000\,12$ and $G/C = 0.439\,62 \pm 0.000\,33$, obtained by Monte Carlo calculations.

The quantity $BC/\sigma a_0$ has been shown in [3] to be universal. These authors expressed their result in terms of $BC/v\sigma$, but they used a definition of p_n as the number of distinct unrooted polygons. As discussed earlier, these results are completely equivalent. It is possible to show directly from the scaling forms of the relevant correlation functions in the $O(N)$ model that C/D is universal. This also follows from [13] and [15], where it was argued that F/C and F/D, respectively, are universal. Hence, the universality of $BC/\sigma a_0$ follows from our result for $BD/\sigma a_0$.

The value of C/D should also be the same for walks on oriented lattices. Since we have argued that in this case $BD/\sigma a_0$ gains an additional factor of two, this should carry over to $BC/\sigma a_0$.

Table 1. Estimates of critical amplitudes $A \ldots J$ defined in the text, for the honeycomb, square, triangular, L and Manhattan lattices. Quantities in parentheses are estimates from the amplitude relations discussed in the text. The remainder are estimates from the corresponding series expansions.

Amplitude	Honeycomb	Square	Triangular	L	Manhattan
A	1.145	1.1771	1.186	1.05	0.89
B	0.6358	0.5623	0.2640	2.47	2.5
C	(0.889)	0.771	0.711	0.67	0.73
D	0.0660	0.05631	0.0518	(0.049)	(0.053)
E	(0.166)	0.1416	0.131	(0.12)	(0.13)
F	(0.125)	0.1082	0.0997	(0.095)	(0.10)
G	(0.389)	0.339	0.313	(0.30)	(0.32)
H	(1.341)	1.272			
J	(0.000972)	(0.002768)	(0.006205)		
μ	1.8477591	2.6381585	4.1507951	1.5657	1.733
σ	2	2	1	4	4
a_0	$3\sqrt{3}/4$	1	$\sqrt{3}/2$	1	1
$32\pi^2 BD/5\sigma a_0$	1.020	1.0000	0.997		
$BC/\sigma a_0$		0.2168	0.2167	0.41	0.46

The amplitude for dumb-bells, J, is related to the amplitude for SAWs A by

$$J = A(1 - 2\tau/\mu + \tau^2/\mu^2)/8 \tag{12}$$

as shown in [16], where $\tau = q - 1$ and q is the lattice coordination number. It was also shown there that the amplitude for lattice trails, H, *on the honeycomb lattice only*, is related to the SAW amplitude by

$$H = 4A/(2 + \sqrt{2}). \tag{13}$$

Details of the various series and their analysis to estimate critical amplitudes have already been discussed in [1]. Since then, extension of the square lattice SAW series [17] and the square lattice trail series [18] have allowed more accurate estimates of these amplitudes. These are given in table 1. Corrected amplitudes for the honeycomb lattice are also given

in table 1. To ensure that the various normalization factors were correct, we generated and analysed the series for the radius of gyratio for triangular lattice and honeycomb lattice polygons, and estimated the radius of gyration amplitude directly. These new series are given in table 2, and the amplitude estimates in table 1. The quantity E(triangular) was given in [1] in units of unit triangles. It is more correctly given in table 1 in units of area, assuming unit lattice spacing as used in the estimate of other metrically dependent amplitudes. Direct estimates of the amplitudes A, B and C for the L and Manhattan lattices are also given. These have been obtained from the series in [19] and [20]. The polygon counts for Manhattan lattice polygons quoted in [19] should be divided by 2 in order that the correct normalization per-site be retained.

Table 2. Number of polygons and the sum of the squares of the radii of gyration of n-step polygons on the triangular and honeycomb lattices.

Triangular			Honeycomb		
n	p_n	$4n^2 p_n \langle R^2 \rangle_n$	n	$2p_n$	$8n^2 p_n \langle R^2 \rangle_n$
3	2	24	6	1	144
4	3	96	8	0	0
5	6	408	10	3	2460
6	15	1872	12	2	3168
7	42	8688	14	12	32 052
8	123	39 912	16	18	77 976
9	380	183 264	18	65	420 444
10	1212	834 744	20	138	1 310 088
11	3966	3 779 064	22	432	5 655 204
12	13 265	17 013 936	24	1074	19 291 968
13	45 144	76 186 320	26	3231	76 066 992
14	155 955	339 566 616	28	8718	268 063 080
15	545 690	1 507 025 568	30	25999	1 011 675 420
16	1 930 635	6 662 739 096			
17	6 897 210	29 355 291 552			
18	24 852 576	12 893 2421592			

The square lattice amplitudes are generally the most accurate. We see that $32\pi^2 BD/5\sigma a_0 = 1.0000$ for the square lattice, 1.020 for the honeycomb lattice and 0.997 for the triangular lattice. This is perfect agreement, given the accuracy of the amplitude estimates, and allows D(L, Man.) to be estimated. Similarly, the invariant $BC/\sigma a_0 = 0.2168$ (square), 0.2167 (triangular) and this value is used to predict C(honeycomb). The ratio $F/C = 0.1403$(square), 0.1402 (triangular), in good agreement with the Monte Carlo estimate cited above. This then permits F(honeycomb, L, Man.) to be estimated. We see that this quantity is a factor 2 greater for the oriented L and Manhattan lattices, as predicted. This factor was not found by Privman and Redner [3], who found that (with their convention for p_n as the number of distinct unrooted oriented polygons) the same result for $BC/v\sigma$ as for unoriented lattices, within their numerical errors. It seems likely that these authors miscounted by a factor of 2. For example, the number of distinct oriented 4-step polygons on the L lattice is 2, not 1.

Similarly, $G/C = 0.4397$(square), and 0.4402(triangular), again in agreement with the precise Monte Carlo estimate. The predicted values of G(honeycomb, L, Man) are given in table 1. The ratio E/D (where E is $E^{(1)}$ in our previous notation) was found to be 2.515(square) and 2.529(triangular), where we believe the square lattice estimate to

be more precise, as the series from which E was estimated is far longer for the square lattice. This value then permits E(honeycomb, L, Man.) to be estimated. The amplitudes J follow from the amplitudes for A and equation (12). The amplitude H(honeycomb) follows from A(honeycomb) and equation (13). All amplitudes quoted are expected to have an associated error confined to the last decimal place quoted. Apart from some minor gaps for the triangular, Manhattan and L lattice amplitude, table 1 now gives a complete and corrected tabulation of critical amplitudes.

Acknowledgments

References

[1]   Enting I G and Guttmann A J 1992 *J. Phys. A: Math. Gen.* **25** 2791
[2]   Cardy J L 1988 *J. Phys. A: Math. Gen.* **21** L797
[3]   Privman V and Redner S 1985 *J. Phys. A: Math. Gen.* **18** L781
[4]   Guttmann A J and Whittington S G 1978 *J. Phys. A: Math. Gen.* **11** 721
[5]   Cardy J L 1988 *Phys. Rev. Lett.* **60** 2709
[6]   Zamolodchikov A B 1986 *Zh. Eksp. Teor. Fiz.* **43** 565; 1986 *(Engl. Trans. JETP Lett.* **43** 730)
[7]   Dotsenko V I and Fateev V A 1984 *Nucl. Phys.* B **240** 312
      Blöte H W, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
[8]   Privman V and Rudnick J 1985 *J. Phys. A: Math. Gen.* **18** L789
[9]   Miller J 1991 *J. Stat. Phys.* **63** 89
[10] Camacho C J and Fisher M E 1990 *Phys. Rev. Lett.* **65** 9
[11] Fisher M E 1992 private communication
[12] Duplantier B 1990 *Phys. Rev. Lett.* **64** 493
[13] Cardy J L and Saleur H S 1989 *J. Phys. A: Math. Gen.* **22** L601
[14] Caracciolo S, Pelisetto A and Sokal A D 1990 *J. Phys. A: Math. Gen.* **23** L969
[15] Privman V, Hohenberg P C and Aharony A 1991 *Phase Transitions and Critical Phenomena* vol 14, ed
          C Domb and J Lebowitz (London: Academic) p 1
[16] Guttmann 1985 *J. Phys. A: Math. Gen.* **18** 567
[17] Conway A R, Enting I G and Guttmann A J 1993 *J. Phys. A: Math. Gen.* **26** 1519
[18] Conway A R and Guttmann A J 1993 *J. Phys. A: Math. Gen.* **26** 1535
[19] Enting I G and Guttmann A J 1985 *J. Phys. A: Math. Gen.* **18** 1007
[20] Malakis A 1973 *J. Phys. A: Math. Gen.* **8** 1885